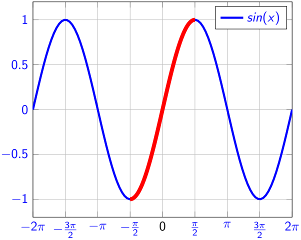
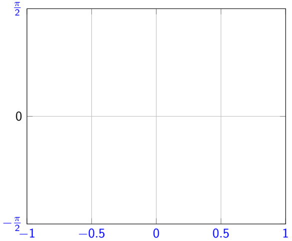
Chapter 3.9: Inverse Trigonometric Functions

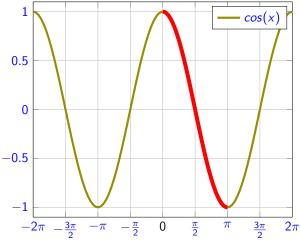
Inverse Trigonometric Functions

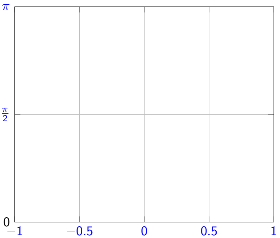




$\sin^{-1}(x) = \arcsin(x)$

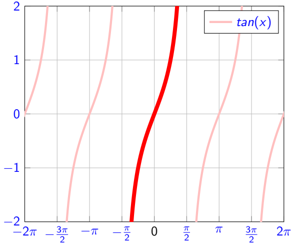
$-1 \leq x \leq 1$

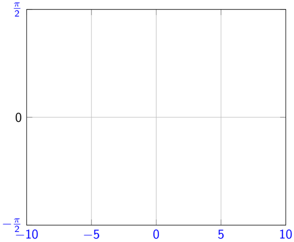




$\cos^{-1}(x) = \arccos(x)$

$-1 \leq x \leq 1$





$\tan^{-1}(x) = \arctan(x)$

$x \in \mathbb{R}$

Derivative of arcsin

1. $\arcsin(-1/2) =$

$$\begin{aligned} x &= \arcsin(-1/2) \\ \sin(x) &= \sin(\arcsin(-1/2)) \\ \sin(x) &= -1/2 \end{aligned}$$

   Solving for x in $[-\pi/2, \pi/2]$ gives $x = -\pi/6$ since $\sin(-\pi/6) = -1/2$.

2. $\cos(\arcsin(x)) =$
   I would love to have $\sin(\arcsin(x))$.
   Recall $\sin(x)^2 + \cos(x)^2 = 1$. Hence
   $\cos(x) = \sqrt{1 - \sin(x)^2}$
   $\cos(\arcsin(x)) =$
   $\sqrt{1 - \sin(\arcsin(x))^2} = \sqrt{1 - x^2}$

$$\frac{d}{dx}[\arcsin(x)] =$$

$$\begin{aligned} y &= \arcsin(x) \\ \sin(y) &= x \\ \frac{d}{dx}[\sin(y)] &= \frac{d}{dx}[x] \\ \cos(y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos(y)} \\ \frac{dy}{dx} &= \frac{1}{\cos(\arcsin(x))} \\ \frac{dy}{dx} &= \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

Derivative of arctan

$$\frac{d}{dx}\arctan(x) =$$

$$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{1}{\cos(x)^2}$$

$$\cos(x)^2 = \frac{1}{\tan(x)^2+1}$$

$$\begin{aligned}\tan(x)^2+1 &= \frac{\sin(x)^2}{\cos(x)^2}+1\\ &= \frac{\sin(x)^2+\cos(x)^2}{\cos(x)^2}\\ &= \frac{1}{\cos(x)^2}\end{aligned}$$

$$\begin{aligned}y &= \arctan(x)\\ \tan(y) &= x\\ \frac{d}{dx}[\tan(y)] &= \frac{d}{dx}[x]\\ \frac{1}{\cos(y)^2}\frac{dy}{dx} &= 1\\ \frac{dy}{dx} &= \cos(y)^2\\ \frac{dy}{dx} &= \frac{1}{1+\tan(\arctan(x))^2}\\ \frac{dy}{dx} &= \frac{1}{1+x^2}\end{aligned}$$

Examples

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

- $\frac{d}{dx}\left[\arccos(x^2)\right] = -\frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$
- $\frac{d}{dx}[\ln(\arctan(x))] = \frac{1}{\arctan(x)} \cdot \frac{1}{1+x^2}$
- $\frac{d}{dx}\left[\arcsin(\sqrt{1-t})\right] = \frac{1}{\sqrt{1-(\sqrt{1-t})^2}} \cdot \frac{1}{2\sqrt{1-t}} \cdot (-1)$
- $\frac{d}{dx}[\arctan(\sin(x))] = \frac{1}{1+\sin(x)^2} \cdot \cos(x)$
- $\frac{d}{dx}\left[\arctan(\sqrt{x})\right] = \frac{1}{1+\sqrt{x}^2} \cdot \frac{1}{2}x^{-1/2}$
- $\frac{d}{dx}\left[\ln(1+x^2)\right] = \frac{1}{1+x^2} \cdot 2x$

More examples for $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

Example: Find the tangent line to $y = \arctan(x)$ at $x = 1$.

Find slope of the tangent line:

$\frac{d}{dx}(\arctan a) = \frac{1}{1+1^2} = \frac{1}{2}$

So we know we will have

$y = \frac{1}{2}x + b$

The y-intercept y can be computed from point $[1, \arctan 1]$.

$\arctan 1 = y$, which gives $1 = \tan y$.

Hence $y = 1/4$.

$\frac{1}{4} = \frac{1}{2} + b$

The final equation of tangent line is

$y = \frac{1}{2}x - \frac{1}{4}$

Example: The poistion of a particle is given by $s(t) = \arctan(t^2)$ where $t \geq 0$. Determine when acceleration is zero.

Well, take the second derivative and solve when it is equal zero.

$v(t) = \frac{d}{dt}\left[\arctan(t^2)\right] = \frac{1}{1+t^4}2t$

$a(t) = \frac{d^2}{dt^2}\left[\arctan(t^2)\right] = \frac{2(1+t^4)-2t(4t^3)}{(1+t^4)^2}$

Now solve

$$a(t) = 0$$
$$0 = \frac{2(1+t^4)-2t(4t^3)}{(1+t^4)^2}$$
$$0 = 2(1+t^4)-2t(4t^3)$$
$$7t^4 = 2$$
$$t = \sqrt[4]{\frac{2}{7}}$$

Chapter 3.9 Recap

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$